The principle of least action and the geometric basis of D-branes

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ABSTRACT: We analyze thoroughly the boundary conditions allowed in classical non-linear sigma models and derive from first principle the corresponding geometric objects, i.e. D-branes. In addition to giving classical D-branes an intrinsic and geometric foundation, D-branes in nontrivial H flux and D-branes embedded within D-branes are precisely defined. A well known topological condition on D-branes is replaced.

KEYWORDS: D-branes, Boundary Quantum Field Theory.

Contents

1. Introduction and conclusion

There have been a great deal of works involving D-branes. Many connect different formalisms and limits using dualities and extrapolate from different branches of mathematics, relating to a diverse range of topics. This reflects the richness of string theory in general but complicates our understanding of D-branes because it is often difficult to pin down a supposed feature of D-brane to any well defined set of assumptions. Very little if any had been done for the foundation of D-branes theory.

The purpose of this paper is to fill this void and carry out a systematic analysis from first principle. Our object of study is the classical non-linear sigma model; our point of departure is the principle of least action [1].[1] Neumann boundary condition for Type I string in flat spacetime has been formulated this way in textbooks [3], and Dirichlet boundary condition was known to be compatible with it [4]. But until now a thorough and general treatment emphasizing logical coherence rather than examples and applications had been absent. We believe such an undertaking is worthwhile given the important roles D-branes currently play in theoretical physics. In this paper we examine the boundary variational problem carefully and thoroughly and solve it in well defined generality. D-brane background is then understood as the collection of data completely characterizing a solution.

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[1]See [2] for a modern introduction

Since our work is foundational rather than applied in nature, it is perhaps surprising that in addition to putting some known aspects of D-branes on a firm ground and explicating their domain of validity, we have also obtained new and even unexpected results. Thanks to the comprehensiveness of our analysis we uncovered solutions not known before. As one of the more striking consequences, a standard argument for a topological condition on D-branes, that the pull-back of the NS 3-form H to it must have vanishing cohomology class, is invalidated. That argument was based on the premise that a D-brane must have a 2-form flux F defined on it whose exterior differential agrees with the pull-back of H. We have shown instead that the data defining a D-brane is a submanifold S and a certain tensor R defined over it, while F does not have to be well defined over S. R has a far more intricate relationship with H that does not lead to an obvious obstruction for the submanifold per se. We give a counterexample of a D-brane that fails the old condition and derive the correct relation between H and backgrounds on D-branes replacing it. The exact nature of D-branes in the presence of H field had always been a puzzle, which is finally solved in this work. Also interesting is the appearance of the embedding of one D-brane within another D-brane as a solution.

While this paper is oriented toward the physics side of D-branes, in the course of this investigation we have also obtained several mathematical results on the geometry of D-branes which will appear in [5] in an appropriately abstract setting. Some of them are announced in this paper.

Our work diverges from other worldsheet studies of D-branes in the literature on the issues of conformal invariance and classical vs. quantum. Our treatment is strictly classical. Although a quantum treatment would certainly be useful, it is usually unfeasible. Much can already be learned from a classical analysis as a limit of the quantum theory. We note also our treatment is directly relevant for the quantum theory because path integral quantization is also based on an action principle.

Conformal invariance is never invoked in this work, but it will be evident that the boundary conditions we derive are automatically compatible with (classical) conformal symmetry, namely breaking it by half. Of course, quantum fluctuation usually renders scale invariance anomalous. As a result, quantum treatment of conformal invariant boundary condition typically employs boundary conformal field theory [6]. While this allows one to consider D-branes in any conformal field theory, including those without a spacetime interpretation or a Lagrangian formulation, it often obscures the fundamental geometric nature of D-branes. On the other hand, while our approach does not ensure conformal invariance after quantization, it extends the geometric notion of D-brane far beyond conform field theories. In fact, our analysis is applicable without change when the bulk Lagrangian in eq. (2.1) is modified by anything not involving X'. Another systematic approach to the problem of (super-)conformal invariant boundary condition considers the geometry of (super-)boundaries on (super-)worldsheet [7].

After D-branes were first constructed as flat extended objects in flat spacetime on which an open string ends [8], one expects this open string worldsheet approach to be generalized to manifolds. Modern treatment began with [9] and [10], which established the widely used worldsheet method for wrapping D-brane around an arbitrary manifold

in arbitrary spacetime. However, subsequent works had not managed to really derive the standard D-brane boundary conditions formulated in [9] from first principle. We do this in section two of this paper. In section three we examine these conditions and find a much more general set of D-brane solutions. In section four we present two examples of these new solutions.

2. Variation

2.1 On $\partial\Sigma$

The 2d classical sigma model on a boundary is described by the action

$$S = \int_\Sigma \left(\frac{1}{2} G(\dot{X}, \dot{X}) - \frac{1}{2} G(X', X') + B(\dot{X}, X') \right) + \int_{\partial\Sigma} i_{\dot{X}} A, \tag{2.1}$$

Here M is the target space manifold and Σ is a smooth 2d manifold with boundary $\partial\Sigma$. $X : \Sigma \to M$. Locally Σ is parameterized by coordinates t and σ. $\dot{X}$ and X' are derivatives with respect to t and σ respectively. G is a nondegenerate metric defined on M. B is a locally defined 2 form whose exterior derivative is a well defined 3-form H on M. $i_{\dot{X}}$ is the insertion operator of $\dot{X}$ acting on covariant tensors of arbitrary rank.

The integral over the boundary is the standard boundary extension of the σ model action, where A is a 1-form whose exact nature we will address presently. It is worth noting that in addition to being explicitly geometric the action eq. (2.1) also possesses conformal symmetry in 2 dimensions.

Because the worldsheet has boundary, the variation of S in the interior contributes a boundary term in addition to that from the boundary action. The solution to the variational problem therefore consists of the equation of motions in the interior of Σ and a set of boundary conditions on $\partial\Sigma$. In this paper we are interested in the latter. Each connected component of the boundary gives rise to an independent condition but they are all of the same structure. So it suffices to consider one such boundary, denoted by $\partial_1\Sigma$. We shall call a boundary condition on such a component a *D-brane boundary condition*. The data characterizing a D-brane boundary condition is called a *D-brane background solution* (to the boundary variational problem) of the σ-model. Variation principle requires

$$\left(G\left(\delta X, X' \right) - i_{\delta X} i_{\dot{X}} e^* B + \delta\left(i_{\dot{X}} A \right) \right)\big|_{\partial_1\Sigma} = 0. \tag{2.2}$$

This is *the* boundary condition required by the principle of least action for eq. (2.1). We consider here solving it by first imposing a condition on

$$\delta X|_{\partial_1\Sigma} \quad . \tag{2.3}$$

δX takes value in $TM|_X$. Because M is an arbitrary manifold, it is not possible to restrict δX without explicitly referring to X. The only way to restrict $X|_{\partial_1\Sigma}$ is to restrict it to a subset S of M. As far as the variational problem is concerned, we should treat separate components of S separately because variation cannot traverse across separated components. Therefore we may assume S is a connected set. But this is not yet enough.

We expect that the end points should be able to move continuously within S. Furthermore we want to perform standard calculus on S so that we can derive equation of motion at the boundary. If we assume nonsingularity, i.e. every point in S has a neighborhood in S that is homeomorphic to the same set, that set must be homeomorphic to $\mathbb{R}^n$ for some positive integer $n \leq d$. It then follows that S is a n dimensional submanifold of M. Note that we have reached this conclusion through very general consideration. This is the first piece of data of a D-brane background solution and we shall say the D-brane is wrapping the submanifold S if it meets certain conditions stated below.

So far we have only considered how to restrict $X|_{\partial_1\Sigma}$, but we really want to restrict $\delta X|_{\partial_1\Sigma}$. Consider a particular point $t \in \partial_1\Sigma$ and let $x = X(t)$. Denote by $\mathcal{W}(x)$ the subspace of $TM|_x$ to which $\delta X(t)$ must belong. Obviously $\mathcal{W}(x) \subset TS|_x$, the subspace of $TM|_x$ tangent to S. It is tempting to think $\mathcal{W}(x)$ must coincide with $TS|_x$. In fact in previous works on the subject this had always been the implicit assumption. We will for the moment again make this assumption but revisit it in the next section. Clearly $\dot{X}$ should be subjected to the same restriction as δX.

Given the above, it is now clear that A exists only on S. And on S it is only locally defined but $F = dA + e^*B$ is well defined over S, where e^* is the pull-back map from the forms on M to those on S. In other words, it is gauge invariant [11], because eq. (2.1) is gauge invariant under $B \to B + d\Lambda$, and $A \to A - e^*\Lambda$. It is also F rather than A which enters the boundary condition: eq. (2.2) can now be rewritten as

$$\left(G\left(\delta X, X'\right) - i_{\delta X} i_{\dot{X}} F\right)\big|_{\partial_1\Sigma} = 0. \tag{2.4}$$

Now it follows that

$$dF = e^*H, \tag{2.5}$$

This is the standard argument leading to the well known topological condition on D-branes:

$$[e^*H] = 0 \tag{2.6}$$

in the De Rham cohomology of S.

Together with $X|_{\partial_1\Sigma} \in S$ this is a complete set of D-brane boundary conditions for the boundary variational problem on $\partial_1\Sigma$. In fact all D-brane boundary conditions for σ-model considered in the literature up to now are of this form, corresponding to a D-brane wrapping the submanifold S with flux F. In the next section we shall show that a lot had been missed.

2.2 Boundary conditions

Let S be an n dimensional submanifold of M. Consider a particular map X, a point[2] $x \in X(\partial_1\Sigma) \subset S$, and a coordinate patch (U, x) in M containing x. With an abuse of notation we use x to also denote the coordination map $U \to \mathbb{R}^d$. We may assume that the first n coordinates also parameterize $S \cap U$ and denote them by $x^A, x^B, \ldots$; the rest are denoted by $x^a, x^b \ldots$. When no distinction between those two groups is desired, greek indices $x^\mu, x^\nu, \ldots$ will be used. We will also choose the coordinates on the worldsheet so that t parameterizes $\partial_1\Sigma$.

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[2] $A \subset B$ means A is a subset of B while $A \subsetneq B$ means A is proper subset of B.

With these notation in mind, eq. (2.4) yields[3]

$$F_{AB}\dot{X}^B + G_{A\mu}X'^\mu = 0;$$
$$\dot{X}^a = 0. \tag{2.7}$$

The second equation stems from the restriction $X|_{\partial_1\Sigma} \in S$.

Now use light-cone derivatives $\partial = \partial_0 + \partial_1$, $\bar{\partial} = \partial_0 - \partial_1$, one finds that the following conditions hold on $\partial_1\Sigma$:

$$(G_1 - F)\bar{\partial}X = (G_1 + F)\partial X + 2G_2\partial X$$
$$\bar{\partial}X^a = -\partial X^a, \tag{2.8}$$

where $G_1 = (G_{AB})$, $G_2 = (G_{Ab})$. We shall assume that $(G_1 - F)$ is invertible, which is guaranteed if G is positive definite. Then eq. (2.8) can be rewritten as

$$\bar{\partial}X^\mu = R^\mu{}_\nu \partial X^\nu, \tag{2.9}$$

where R is given by

$$\begin{pmatrix} (G_1 - F)^{-1}(G_1 + F) & 2(G_1 - F)^{-1}G_2 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{2.10}$$

Note that R is nondegenerate and like F has S as its domain.

Although we have written R explicitly in components, it can be also be written more abstractly [5]. Thus R is a well defined section of the pull-back of the bundle of $(1,1)$ tensors on M to S. An important property of R is[4]

$$R^T G R = G \tag{2.11}$$

if the pull-back of G on S is nondegenerate. Here we give a proof in explicit components. If G_1 is nondegenerate, $\mathcal{W}(x)$ has an orthogonal complement with respect to G. So $\forall x \in S$, one can find some coordinate patch containing x so that G_2 vanishes at x. There both R and G are block diagonal, as are the two sides of eq. (2.11). The lower right block of the equation holds trivially. For the upper left block, consider[5] $\beta^A{}_D \equiv G^{AC}B_{CD}$. It satisfies $\beta^T G = -G\beta$. Then $(\mathbb{I} \pm \beta^T)G = -G(\mathbb{I} \mp \beta)$. The upper left block of R is $(1 - \beta)^{-1}(1 + \beta)$. Eq. (2.11) then follows.

The following is therefore also a complete set of boundary condition for the variation problem of eq. (2.1).

$$X|_{\partial_1\Sigma} \in S;$$
$$R\,\partial X = \bar{\partial}X. \tag{2.12}$$

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[3]Summation over repeated indices is implicit unless stated otherwise.

[4]Note this is precisely the requirement for the boundary condition to preserve half of the conformal invariance [9].

[5]In accordance with custom, $G^{AC} \equiv (G^{-1})^{AC}$.

3. Analysis

Because we have derived the second of eq. (2.12) from eq. (2.7), the two might appear to be equivalent, between which one can choose either R or F as characterizing the background on the D-brane. But this is not quite right. The R obtained from the F in the last section is subject to a constraint: the multiplicity of its eigenvalue at -1 is always $d - n$. Although this might appear natural at first, it does not seem strictly necessary. If we relax the condition on R and still find a boundary condition, it will be outside the domain of validity of eq. (2.7). In this section we examine this possibility thoroughly.

3.1 D-branes old and new

Consider a pair (S, R), where S is a n dimensional submanifold M and R is a section of the pull-back of the bundle of fiberwise endomorphisms over TM to S. We require eq. (2.11), which implies that R is diagonalizable. Define $P = R + 1$. Consider a neighborhood N of a point $x \in S$. It is clear that $\dot{X} \in \mathcal{W} \equiv \mathrm{Im} P$. For this to be consistent with the first of eq. (2.12) we must require $\mathcal{W}(x) \subset TS|_x$ so $\mathrm{rank}\, P \leq n$. The known cases correspond to $\mathrm{rank}\, P = n$ and $\mathcal{W} = TS|_x$. This still leaves the possibility of $\mathcal{W} \subsetneq TS|_x$.

Suppose $\mathrm{rank}\, P = n - k$. So we may diagonalize R at x with k eigenvectors with eigenvalue -1. Then R is of the form eq. (2.10), with the lower right block of dimension k. This can be shown by reversing the steps below eq. (2.11). k is clearly upper semi-continuous on S, i.e. it cannot increase suddenly. So without loss of generality we may assume k does not increase over N. Of course k may very well decrease in N. Consider the connected component containing x of the set $N_k \equiv \{x \in N \,|\, \mathrm{rank}\, \mathcal{W}(x) = n - k\}$. In light of the consideration in §2.1, this is a valid boundary condition if N_k is itself a submanifold of codimension k and $\forall x \in N_k,\ \mathcal{W}(x) = TN_k$.

The case $\mathrm{rank}\, P < n$ is entirely new, because it is not translatable to the pair (S, F) describing D-brane boundary conditions with a background gauge field. However, it has a very intuitive interpretation both physically and mathematically. If the submanifold N_k exists, on the basis of our early analysis it can and must be interpreted on its own as a neighborhood of a $n - k$ dimensional (connected) D-brane, which we call S_k. Of course a given D-brane S with background R may contain several such D-branes. Here we just consider one of them. The boundary conditions on S_k are prescribed by the pair $(S_k, R|_{S_k})$ which on its own provide a complete set of boundary conditions for the variational problem of eq. (2.1). However, it is also a part of the boundary conditions specified by (S, R) in the precise mathematical sense just described. Therefore we have a D-brane S_k with background $R|_{S_k}$ embedded in the bigger D-brane S with background R. Furthermore, by reapplying the analysis above, one sees that S_k itself may contain lower dimensional D-branes, say some $(S_k)_l$ in an obvious extrapolation of previous notation. The latter can also be interpreted directly as S_{k+l}. Therefore we can consider nested structure of D-branes within D-branes as submanifolds contained within submanifolds with background R defined on them through restriction. The submanifold structure is arbitrary but entirely determined by the background R. Define $S_k^\cup \equiv \{x \in S \,|\, \mathrm{rank}\, \mathcal{W}(x) \leq n - k\}$. $S_k^\cup$ is the

union of all possible S_k's. We clearly have the following filtration:

$$S_n^\cup \subset \ldots S_1^\cup \subset S_0^\cup = S. \qquad (3.1)$$

There is an analytical constraint on the filtration. R is nondegenerate, so the sign of its determinant cannot vary continuously over S. R is real, so its eigenvalues are either real or in complex conjugate pairs. R is orthogonal, so they are either $e^{\pm i\theta}$ or ± 1. It follows then the parity of the multiplicity of the eigenvalue -1 of R cannot vary continuously over S. When considering continuously varying R, we need to consider only S_k for even k. In such cases, the embedded branes always have even codimensions.

3.2 Physical interpretation

While we have tried to avoid excessive abstraction, so far our emphasis is analytical. Yet what we found has a very direct and clear physical interpretation, even though the context is a little surprising. D-branes, after all, are dynamical extended objects in superstring theory. It is well known that in superstring theory, a Dp-brane can carry RR charges for the D(p-2)-brane [12], thus forming a bound state. This structure can be nested, and the codimensions may be any even number bounded by p. This is precisely what we have found! However, there are some very intriguing aspects of our results. The identification of embedded branes in superstring theory required various dualities [13], [12], [14] and the relation between their RR charge and the gauge field on the D-branes was understood from anomaly cancellation for the D-brane worldvolume theory, which contains chiral fermions [15][6][16]. All those arguments are only applicable to superstring theories. The analysis in this paper can be generalized to the supersymmetric *sigma* model [5], and the same phenomenon obviously persists So our results agree well with the known results and provides the first *exact* description of a localized embedded D-brane. Because our analysis is carried out from first principle and independent of any duality argument, it can be seen as providing indirect support for the validity of several string dualities. What is the most striking, however, is that the domain of validity of our analysis is much more general than superstring theories and applicable *directly* to any σ-model type action as mentioned in the introduction, suggesting that this phenomenon is universal.

It has been long observed that the background F interpolates between Neumann and Dirichlet boundary conditions. So if the component of F over a plane is taken to infinity then in some sense these directions become Dirichlet because they will have -1 eigenvalue, indicating a "smaller" brane. The hierarchy of $S_k^\cup$ we find above finally give this intuitive idea a concrete and mathematically precise realization.

3.3 General solutions

It turns out that there are a lot more new D-brane background solutions besides the embedded branes discussed above. Rather than examining them case by case, we take

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[6]Note: while it was correct in using anomaly inflow to deduce the RR coupling of D-branes, [15] missed an important part of brane anomalies as well as a key subtlety in the action of RR fields. See [16] for a detailed general treatment. The differences have profound implications for the physics and mathematics of D-branes.

a general ground and look for the *most* general of D-brane boundary conditions for the sigma model. This entails finding the most general form of the boundary action, because a variational problem is determined by an action.

We now describe a most natural generalization of the boundary part of eq. (2.1) that we have discovered. We will emphasize the intuitive and conceptual aspects relevant for the physical problem and a physicist's viewpoint, while reserving the detail of the formal development and generalization for [5]. Let $\mathcal{W}$ be a general distribution on S such that every open set in S contains a point x at which $\mathcal{W}(x) = TS|_x$. We say $\mathcal{W}$ is a *mannequin* for S. By this definition, there may be a nonempty set of points where $\operatorname{rank}\mathcal{W}$ is not at a local maximum, but its complement must be dense in S.

Consider now the dual $\mathcal{W}^*$ in T^*M: $\forall x \in S$, $\mathcal{W}^*(x) = (\mathcal{W}(x))^*$. Sections of p-th exterior powers of $\mathcal{W}^*$ will be called $\mathcal{W}$ p-forms. The inclusion of $\mathcal{W}$ in TS induces a pull-back map $e^*_{\mathcal{W}}$ from p-forms on S to $\mathcal{W}$ p-forms. We require $\mathcal{W}$ to be involutive, i.e. its sections are closed under Lie bracket. One can therefore define a complex of $\mathcal{W}$ p-forms in the usual fashion with the following exterior differential.

$$
d\omega(V_0, \ldots, V_n) \equiv \sum_{i=0}^{n} (-1)^i \{ L_{V_i} \, \omega(\ldots, V_{i-1}, V_{i+1}, \ldots)
$$
$$
- \sum_{j=i+1}^{n} \omega(\ldots, V_{i-1}, V_{i+1}, \ldots, V_{j-1}, [V_i, V_j], V_{j+1}, \ldots)\}. \tag{3.2}
$$

Then $e^*_{\mathcal{W}}$ is a chain map.

We are now ready to write down the general boundary action. Given a submanifold S of M, a mannequin $\mathcal{W}$ of S, and a locally defined $\mathcal{W}$ 1-form A, consider the following replacement for the boundary part in eq. (2.1):

$$
\int_{\partial_1 \Sigma} i_{\dot{X}} A. \tag{3.3}
$$

The total action is gauge invariant under $B \to B + d\Lambda$, and $A \to A - e^*_{\mathcal{W}} \circ e^* \Lambda$. Therefore $F \equiv dA + e^*_{\mathcal{W}} \circ e^* B$ is a well defined gauge invariant $\mathcal{W}$ 2-form. With the condition

$$
X|_{\partial_1 \Sigma} \in S;
$$
$$
\delta X|_{\partial_1 \Sigma}, \ \dot{X}\Big|_{\partial_1 \Sigma} \in \mathcal{W}(X) \tag{3.4}
$$

variational principle then leads to the following equation

$$
e^*_{\mathcal{W}} \circ e^* \circ i_{X'} G - i_{\dot{X}} F\big|_{\partial_1 \Sigma} = 0. \tag{3.5}
$$

By the same analysis as in §2.2, we obtain again a matrix R relating $\bar{\partial}X$ to ∂X when the nondegeneracy conditions stated there holds. This shows that pair $(\mathcal{W}, F)$ is in fact in 1-1 correspondence with R under those conditions. A more abstract derivation is also possible[5].

As before it is F rather than A that enters in the boundary condition. The condition for F is clearly

$$
dF = e^*_{\mathcal{W}} \circ e^* H \tag{3.6}
$$

This replaces eq. (2.5). This change however is not a mere generalization but instead carries very different meaning and significant ramification. Eq. (2.6) is a topological condition on H and the submanifold S, and when H has nontrivial cohomology class in M, there are some submanifolds S which are excluded by this condition. By contrast, eq. (3.6) depends on the choice of an involutive mannequin, which seems quite arbitrary. Even if a submanifold S is excluded by eq. (2.6), as long as we can find an involutive mannequin $\mathcal{W}$ *compatible* with H in the sense that

$$[e_{\mathcal{W}}^* \circ e^* H] = 0 \tag{3.7}$$

in the d-cohomology of $\mathcal{W}$ forms, our analysis shows that a D-brane can still wrap around S. In §4.2 we will present an instance of a submanifold which eq. (2.6) says cannot be wrapped by D-brane but which admits an exotic D-brane background solution of the type that we have discovered. Therefore eq. (2.6) is invalidated. Does eq. (3.7) imply a more lenient condition on S or simply removes any? While we do not yet have a proof, based on observational evidence we conjecture that any manifold can be wrapped by D-brane in the context of classical σ-model. Namely, for any given H and S, we conjecture there is some involutive mannequin for S compatible with H.

In summary, for a spacetime triple (M, G, H), where M is a differential manifold, G a nondegenerate metric on M, and H a closed 3-form on M. A D-brane background solution consists of $(S, \mathcal{W}, F)$, where S is a submanifold in M, $\mathcal{W}$ an involutive mannequin for S, and F a $\mathcal{W}$ 2-form satisfying eq. (3.6). The mannequin $\mathcal{W}$ is said to wear the D-brane on S with background F.

4. Examples

In this section we consider two exotic examples of D-branes background solution not known before.

4.1 D2-brane wrapping S^2 with a D0-brane embedded

Consider a D2-brane having the topology of S^2. We shall work in spherical coordinates and use the $O(3)$ symmetric metric $ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$. Consider the involutive mannequin given by the linear hull of

$$\begin{aligned} \partial_\phi \ , \\ \cos\frac{\theta}{2}\sin\phi \, \partial_\theta \, + \, \frac{\cos\theta\cos\phi}{2\sin\frac{\theta}{2}} \, \partial_\phi, \\ \cos\frac{\theta}{2}\cos\phi \, \partial_\theta \, - \, \frac{\cos\theta\sin\phi}{2\sin\frac{\theta}{2}} \, \partial_\phi \end{aligned} \tag{4.1}$$

everywhere and the background

$$F = 2\sin^2\frac{\theta}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.2}$$

This corresponds to the following R background

$$\begin{pmatrix} \cos\theta & -\sin^2\theta \\ 1 & \cos\theta \end{pmatrix} \tag{4.3}$$

R is smooth everywhere and develops two Dirichlet directions at the south pole, indicating that a D0-brane is embedded. F is defined everywhere but there.

4.2 D-brane on S^3 with nontrivial H

The simplest manifold which the supposed topological condition eq. (2.6) would have prohibited a D-brane from wrapping is an S^3 over which there is a H field of nonvanishing De Rham cohomology.

Employ the Hopf coordinates $\theta/2$, ϕ, and ρ, so that the volume form is $H = \sin\theta d\theta d\phi d\rho$ and ρ is the coordinate for the circle fiber. Consider the 2-form

$$F = 2\sin^2\theta/2 \, d\phi \, d\rho. \tag{4.4}$$

It becomes singular at the south pole but everywhere else its differential is the volume form. This background F is well defined for the involutive mannequin of S^3 that is everywhere the linear hull of

$$\partial_\phi, \quad \partial_\rho,$$
$$\cos\frac{\theta}{2}\sin\phi \, \partial_\theta + \frac{\cos\theta\cos\phi}{2\sin\frac{\theta}{2}} \, \partial_\phi.$$
$$\cos\frac{\theta}{2}\cos\phi \, \partial_\theta - \frac{\cos\theta\sin\phi}{2\sin\frac{\theta}{2}} \, \partial_\phi. \tag{4.5}$$

Therefore H satisfy eq. (3.6) and hence D-brane can wraps this S^3.

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